Surgery, quantum cohomology and birational geometry

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1 Introduction

Recently, some amazing relations between quantum cohomology and birational geometry have been discovered. Surgeries play a fundamental role in these recent discoveries. This expository article surveys these new relations.

In the early days of quantum cohomology theory, the author [R3] showed that Mori's extremal ray theory can be generalized to symplectic manifolds from projective manifolds. This gave the first evidence of the existence of a link between quantum cohomology and birational geometry. Subsequently, attention was diverted to establishing a mathematical foundation for quantum cohomology. The progress on this link was slow. There are several papers ([Wi2],[Wi3] and others) extending results of [R3] to Calabi-Yau 3-folds and crepant resolutions. It is time to think these mysterious connections again. The breakthrough appeared in the paper of Li-Ruan [LR] where they calculated the change of quantum cohomology under flops and small extremal transitions for 3-folds. Their results showed a surprising naturality property of quantum cohomology under these surgeries. At present, this naturality property is best understood in complex dimension three. The scope of this new naturality property is still unknown. The author believes that there is some very interesting mathematics hidden in these intrigue properties and that the general machinery of quantum cohomology can be very useful in uncovering them.

One of the goals of this paper is to attract more people to work on this topic. Hence, the author shall try to make some conjectures and proposals for general cases to entertain the readers. Since it is a survey instead of a research paper, the author will concentrate on the ideas and techniques. The main reference of this article is Li-Ruan's paper [LR]. The reader can find more details and more complete references there. The author apologizes for the lack of details.

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This paper is organized as follows. We give a brief survey on quantum cohomology and the naturality problem in section 2 and fix notation. Then, we will briefly outline Mori's minimal model program in birational geometry. We will discuss the relation between the naturality problem and birational geometry in section 4. The main theorem was proved by a degeneration technique, developed independently by Li-Ruan [LR] and Ionel-Parker [IP]. This survey presents only Li-Ruan's approach in section 5. We will discuss the possible generalization of Mori's program in the last section. The author would like to thank H. Clemens for explaining to him the semi-stable degenerations. The author benefited from many interesting conversations with J. Kollár, An-Min Li and D. Morrison on the topics related to this article. He wishes to express his thanks to them.

2 Quantum cohomology and its naturality

Due to the efforts of many people, the mathematical theory of quantum cohomology is well-understood. We refer to [R6] for a survey. Developing the theory was rather difficult and time-consuming. Naturally, we are looking for some applications to justify the time and energy we spent in building the general machinery. If we expect quantum cohomology to be as useful as cohomology, the results so far are disappointing.

The fundamental reason that cohomology is very useful is its naturality. Namely, a continuous map induces a ring homomorphism on cohomology. A fundamental problem in quantum cohomology is the

Quantum naturality problem: Define "morphism" for symplectic manifolds so that quantum cohomology is natural.

The author believes that the understanding of the naturality of quantum cohomology will be essential for the future success of quantum cohomology theory.

It has been known for a while that quantum cohomology is not natural, even with respect to holomorphic maps. A crucial calculation is the quantum cohomology of projective bundles by Qin-Ruan [QR]. The calculation clearly demonstrated that quantum cohomology is not natural for fibrations. This lack of naturality is very different from ordinary cohomology and severely limits our efforts to develop some nice consequences of cohomology theory, such as characteristic classes. The Qin-Ruan result shows that possible "morphisms" must be very rigid. The existence of these rigid morphisms will set apart quantum cohomology from cohomology and give it its own identity. The author does not know the full story at present. In this article, we will describe a class of

"morphisms" of symplectic manifolds related to birational geometry. This class of "morphisms" is a certain class of surgeries called "transitions".

Let us briefly review quantum cohomology and naturality. We start with the definition of GW (Gromov-Witten) invariants. Suppose that (M, ω) is a symplectic manifold of dimension 2n and J is a tamed almost complex structure. Let $\overline{\mathcal{M}}_A(J,g,k)$ be the moduli space of stable J-holomorphic genus g maps with fundamental class A and k-marked points. There are several methods of defining GW-invariants ([FO], [LT3], [R5], [S]). One method ([R5],[S]) is to construct a "virtual neighborhood" (U_k, S_k, E_k) with the following properties: (1) U_k is a smooth, oriented, open orbifold whose points are maps; (2) E_k is a smooth, oriented orbifold bundle over U_k ; (3) S_k is a proper section of E_k such that $S^{-1}(0) = \overline{\mathcal{M}}_A(J,g,k)$. There is a map

$$\Xi_k: U_k \to M^k$$

given by evaluating a map at its marked points. Let Θ be a Thom form of E_k supported in a small neighborhood of the zero section. We define

(2.1)
$$\Psi_{(A,g,k)}^{M} = \int_{U_k} S_k^* \Theta \wedge \Xi_k^* \prod_i \alpha_i$$

for $\alpha_i \in H^*(M, \mathbf{R})$. One can eliminate divisor classes $\alpha \in H^2(M, \mathbf{R})$ by the relation

(2.2)
$$\Psi^{M}_{(A,g,k+1)}(\alpha,\alpha_1,\cdots\alpha_k) = \alpha(A)\Psi^{M}_{(A,g,k)}(\alpha_1,\cdots,\alpha_k),$$

for $A \neq 0$.

The above invariants are only *primitive* GW-invariants. In general, we can also pull back cohomology classes of $\overline{\mathcal{M}}_{g,k}$ and insert them into the formula (2.1). But these more general invariants are conjectured ([RT2]) to be computed by primitive invariants. When $A=0, g=0k\neq 3$, we define $\Psi^M_{(A,0,k)}=0$ and $\Psi^M_{(0,0,3)}(\alpha_1,\alpha_2,\alpha_3)=\alpha_1\cup\alpha_2\cup\alpha_3[M]$.

Choose a basis A_1, \dots, A_k of $H_2(M, \mathbf{Z})$. For $A = \sum_i a_i A_i$, we define the formal product $q^A = (q_{A_1})^{a_1} \cdots (q_{A_k})^{a_k}$. For each cohomology class $w \in H^*(M)$ we define a quantum 3-point function

(2.3)
$$\Psi_w^M(\alpha_1, \alpha_2, \alpha_3) = \sum_k \frac{1}{k!} \sum_A \Psi_{(A,0,k+3)}^M(\alpha_1, \alpha_2, \alpha_3, w, \dots, w) q^A,$$

where w appears k times. Here, we view Ψ^M as a power series in the formal variables $p_i = q^{A_i}$. Clearly, a homomorphism on H_2 will induce a change of variables p_i . To define the quantum product, we need to fix a symplectic class $[\omega]$ to define q^A as an element of the Novikov ring Λ_{ω} (see [RT1]). Formally, we can define quantum multiplication by the formula

(1.8)
$$\alpha \times_{\mathcal{O}}^{w} \beta \cup \gamma[M] = \Psi_{w}^{M}(\alpha, \beta, \gamma).$$

To define a homomorphism of the quantum product, we need to match Novikov rings and hence symplectic classes. Here, we view quantum cohomology as a theory of Gromov-Witten invariants instead of just the quantum product. Since the GW-invariants are invariant under deformations, the symplectic class is not a fundamental ingredient of the quantum cohomology. In fact, the symplectic class often obstructs our understanding of quantum cohomology. Clearly, the quantum 3-point function contains the same information as the quantum product. It is more convenient to work directly with the quantum 3-point function.

Definition 2.1: Suppose that

$$\varphi: H_2(X, \mathbf{Z}) \to H_2(Y, \mathbf{Z}), \ H^{even}(Y, \mathbf{R}) \to H^{even}(X, \mathbf{R})$$

are group homomorphisms such that the maps on H_2, H^2 are dual to each other. We say φ is natural with respect to (big) quantum cohomology if $\varphi^*\Psi_0^X = \Psi_0^Y$ ($\varphi^*\Psi_{\varphi^*w}^X = \Psi_w^Y$) after the change of formal variable $q^A \to q^{\varphi(A)}$. If φ is also an isomorphism, we say φ induces an isomorphism on (big) quantum cohomology or X and Y have the same (big) quantum cohomology.

We treat two power series F, G the same if F = H + F', G = H + G' where G' is an analytic continuation of F'. For example, we can expand $\frac{1}{1-t} = \sum_{i=0} t^i$ at t = 0 or $\frac{1}{1-t} = \frac{1}{-t(1-t^{-1})} = -\sum_{i=0} t^{-i-1}$ at $t = \infty$. Hence, we will treat $\sum_{i=0} t^i, \sum_{i=0} t^{-i-1}$ as the same power series.

When X, Y are 3-folds, such a φ is completely determined by maps on H_2 . For example, the dual map of $\varphi: H_2(X, \mathbf{Z}) \to H_2(Y, \mathbf{Z})$ gives a map $H^2(Y, \mathbf{R}) \to H^2(X, \mathbf{R})$. A map $H^4(Y, \mathbf{R}) \to H^4(X, \mathbf{R})$ is Poincaré dual to a map $H_2(Y, \mathbf{R}) \to H_2(X, \mathbf{R})$. In the case of a flop, the natural map $H_2(X, \mathbf{Z}) \to H_2(Y, \mathbf{Z})$ is an isomorphism. Therefore, we can take the map $H_2(Y, \mathbf{R}) \to H_2(X, \mathbf{R})$ as its inverse.

GW-invariants are invariants under symplectic deformation. Hence, two symplectic deformation equivalent manifolds have isomorphic big quantum cohomology. Holomorphic symplectic varieties are such examples. But these are trivial examples. The only known nontrivial examples are given by the work of Li-Ruan [LR] (see section 4).

From the physical point of view, it is also natural (perhaps better) to allow certain **Mirror Transformations**. The author does not know precisely what are these mirror transformations, but they appear naturally in mirror symmetry and in the quantum hyperplane section conjecture [Kim]. From known examples, they must include a nonlinear changes of coordinates of $H^*(X, \mathbf{R}), H^*(Y, \mathbf{R})$ and a scalings of the 3-point function. Readers can find more motivation for mirror transformations in the next section.

Definition 2.2: Under the assumptions of Definition 2.1, we say φ is p-natural with respect to (big) quantum cohomology if we allow a mirror transformation in Definition 2.1. In the same way, we say φ is a p-isomorphism of (big) quantum cohomology if φ is an isomorphism and p-natural with respect to (big) quantum cohomology.

It is obvious that naturality (isomorphism) implies p-naturality (p-isomorphism). The author does not know any nontrivial example of p-naturality or p-isomorphism at present.

Another interesting formulation is the notion of Frobenius manifold due to B. Dubrovin [D]. Here, we require that the bilinear form $\langle \alpha, \beta \rangle = \int_X \alpha \wedge \beta$ be preserved under φ . This is the case for blow-downs. But the intersection form may not be preserved under general transitions when we have nontrivial vanishing cycles (see next section). All three constructions (quantum products, Frobenius manifolds, quantum 3-point functions) contain the same quantum information. Their difference lie in the classical information such as the symplectic class and bilinear form, which can be studied separately. The author believes that Frobenius manifolds may be useful in understanding mirror transformations.

3 Birational geometry

In this section, we review Mori's minimal model program. There are several excellent reviews on this topic [K2],[K3]. We shall sketch the ideas mainly for non-algebraic geometers. In the process, we introduce the needed surgeries we need. Birational geometry is a central topic in algebraic geometry. The goal of birational geometry is to classify algebraic varieties in the same birational class. Two projective algebraic varieties are birational to each other iff there is a rational map with a rational inverse. We call this map a birational map. A birational map is an isomorphism between Zariski open sets. Note that a birational map is not necessarily defined everywhere. If it is defined everywhere, we call the map a contraction. By definition, a contraction changes a lower dimensional subset only. Hence, we can view a contraction as a surgery. Intuitively, a contraction simplifies a variety. For algebraic curves, the birational equivalence is the same as the isomorphism. In dimension two, any two birational algebraic surfaces are related by blow-up and blow-down. The blow-down is a contraction. If the surface is neither rational nor ruled, one can always perform blow-down until reaching a "minimal model" where blow-down can not occur. This is the reason that Mori's program is often called minimal model program. We want to factor a birational map as a sequence of contractions and find the minimal ones in the same birational class.

In two dimensions, the minimal model is unique. In higher dimension, it is much more difficult to carry out the minimal model program. The first difficulty is that the contraction is not unique. To overcome this difficulty, Mori found a beautiful correspondence between contractions and some combinatorial information of the algebraic manifold itself. This correspondence is Mori's extremal ray theory.

To define extremal rays, consider Mori's effective cone

(3.1).
$$\overline{NE}(X) = \{ \sum_{i} a_i A_i, a_i \ge 0 \ A_i \text{ is represented by a holomorphic curve} \} \subset H_2(X, \mathbf{R})$$

By definition, $\overline{NE}(X)$ is a closed cone. Let K(X) be the ample cone. One of the nice properties of projective geometry is

$$(3.2) \overline{NE}(X) = \overline{K(X)^*}.$$

In symplectic geometry, we do not know if this property is true. This lack of knowledge is one of the primary difficulties of symplectic geometry. An edge of $\overline{NE}(X)$ is called an extremal ray of X. Suppose that L is an extremal ray with $K_X \cdot L < 0$. Mori showed that an extremal ray is represented by rational curves. Each extremal ray L gives a nef class H_L such that $H_L \cdot C = 0$ if $[C] \in L$ and $H_L \cdot C > 0$ if $[C] \notin L$, where C is a holomorphic curve. In general, a class H is nef iff $H \cdot C \geq 0$ for any holomorphic curve C. Then H_L defines a contraction $\phi_L : X \to Y$ by contracting every curve whose homology class is in L. Moreover, the Picard number has relation Pic(X) = Pic(Y) + 1. ϕ_L is called a primitive contraction. We abuse the notation and say that ϕ_L contracts L. One can contract several extremal rays simultaneously. This contraction corresponds to contracting an extremal face or other higher dimensional boundary. Conversely, if we have a contraction $\phi: X \to Y$ such that Pic(X) = Pic(Y) + 1, then $\phi = \phi_L$ for some extremal ray.

Once the contraction is found, an additional difficulty arises because Y could be singular even if X is smooth. Obviously, the topology of Y is simpler. However, if the singularities of Y become more and more complicated, we just trade one difficult problem for another equally difficult problem. The next idea is that we can fix the class of singularities. There are two commonly used classes of singularities: terminal singularities and canonical singularities. We refer readers to [K2] for the precise definitions. The terminal singularity is the minimal class of singularities in the minimal model theory. It was shown that in the complex dimension three, the minimal model program can indeed be carried out. Other than for a well-understood exceptional class of varieties (uniruled varieties), the minimal model exists. A minimal model is, by definition, an algebraic variety X such that (i) X has terminal singularities only, and (ii) the canonical bundle K_X is nef.

One should mention that contractions are not enough to find the minimal model. A more difficult operation "flip" is need. We will not give any details about flip. However, its companion "flop" is very important to us because it resolves another difficulty of the minimal model program. Unlike the case in dimension two, the minimal model is not unique in dimension three or more. However, different minimal models are related by a sequence of flops [Ka], [K1].

We now give a local description of a "simple flop". Let Y_s be a three-fold with a ordinary double points. Namely, the neighborhood of a singular point is complex analytically equivalent to a hypersurface of \mathbb{C}^4 defined by the equation

$$(3.3) x^2 + y^2 + z^2 + w^2 = 0.$$

The origin is the only singular point. When we blow up the origin to obtain Y_b , we obtain an exceptional divisor $E \cong \mathbf{P}^1 \times \mathbf{P}^1$. Then, we contract one ruling to obtain Y containing a rational curve with normal bundle O(-1) + O(-1). We can also contract another ruling to obtain \tilde{Y} containing a rational curve with normal bundle O(-1) + O(-1). Clearly, Y, \tilde{Y} are the same locally. However, they could differ globally. The process from Y to \tilde{Y} is called a simple flop. For smooth threefolds, a general flop can be deformed locally to the disjoint union of several simple flops.

A final difficulty is that Mori's minimal model program has only been carried out for complex dimension three. It is still an open problem for higher dimensions.

The fundamental surgeries in birational geometry are contractions, flips, and flops. Every smooth Calabi-Yau 3-fold is a minimal model by definition. For minimal models, we only have flops. But flops are not enough to classify smooth Calabi-Yau 3-folds. We need to introduce another surgery called *extremal transition* or *transition*. As we mentioned previously, a contraction could lead to a singular manifold. Sometimes, a contraction could construct a manifold with singularities beyond the class of terminal singularities. In fact, the flip was introduced precisely to deal with this problem, where it will improve the singularity. The smoothing is another well-known method to improve the singularity. A smoothing is as follows. Consider

(3.4)
$$\pi: U \to D(0, \epsilon) \subset \mathbf{C}$$

where U is an analytic variety and π is holomorphic and of maximal rank everywhere except at zero. Then $X_z = \pi^{-1}(z)$ is smooth except at the central fiber X_0 . In this situation, we say that X_z is a smoothing of X_0 . If π is also of maximal rank at zero, X_0 is smooth and X_z is deformation equivalent to X_0 . Hence, they have the same quantum cohomology. If U is also Kähler, we say

 (U,π) is a Kähler smoothing. A transiton is a composition of a contraction and a Kähler smoothing. We call a contraction small if the exceptional locus is of complex codimension two or more. We call a transition small if its corresponding contraction is small. A flip-flop is a small operation in the sense that it only changes a subset of codimension two. Incidentally, flip-flop has been completely classified in dimension three. In higher dimensions, the classification is still an open question. It was conjectured that any two Calabi-Yau 3-folds can be connected to each other by a sequence of flops or transitions and their inverses.

We first study the change of topology regarding flops and transitions, which is easy in dimension three. Suppose that $F: X \leadsto \tilde{X}$ is a flop, where X, \tilde{X} are 3-folds. In this case, the exceptional locus is of complex dimension one. Each $A \in H_2(X, \mathbf{Q})$ is represented by a pseudo-submanifold Σ . Using PL-transversality, we can assume that Σ is disjoint from the exceptional locus. Then Σ can also be viewed as a pseudo-submanifold of \tilde{X} . We can also reverse this process. Therefore, F induces an isomorphism on $H_2(X, \mathbf{Z})$ and hence $H^2 = Hom(H_2, \mathbf{Q})$. Using Poincaré duality, it induces an isomorphism on H^4 as well. The maps on H^0, H^6 are obvious. The first important theorem is

Theorem 3.1 (Li-Ruan): Under the previous assumptions, F induces an isomorphism on quantum cohomology. In particular, any two three-dimensional smooth minimal models have isomorphic quantum cohomology, where the isomorphism is induced by flops.

In dimension ≥ 4 , flop is not completely understood. However, there are theorems of Batyrev [Ba] and Wang [Wa] that any two smooth minimal models have the same Betti number. We conjecture that

Quantum Minimal Model Conjecture: Any two smooth minimal models in any dimension have isomorphic quantum cohomology.

The Quantum Minimal Model Conjecture is true for holomorphic symplectic varieties as well. In this case, two birational holomorphic symplectic varieties are deformation equivalent. Hence, there is an abstract isomorphism on big quantum cohomology. However, it is still an interesting question to give a geometric construction of the isomorphism.

For transitions, let $c: X \to X_s$ be the contraction and $\pi: U \to D(0, \epsilon)$ be the Kähler smoothing such that X_s is the central fiber. There is a deformation retract $r: U \to X_s$. Therefore, there is a map $\delta: H^*(X_s, \mathbf{Q}) \to H^*(X_z, \mathbf{Q})$, where $X_z = \pi^{-1}(z)$ is the nearby smooth fiber. In the case

of Kähler smoothing, the image of δ can be described in terms of monodromy. Readers can find more detailed references in [C]. The result is as follows. We first blow up the singular points of U along the central fiber to obtain a smooth complex manifold V whose central fibers are a union of smooth complex submanifolds intersecting transversally with each other. We then perform base change (pull-back by a map $z \to z^k$ on the base) to obtain W. W has the additional property that every component of the central fiber has multiplicity 1 in the sense that the function

$$\tilde{\pi}: W \to V \to U \to D(0, \epsilon)$$

vanishes to order one on each component. This process is called *semi-stable reduction*. A consequence of semi-stable reduction is that the monodromy r of $\tilde{\pi}$ is unipotent. In this case, we can define

(3.6)
$$\log(r): H^*(X_z, \mathbf{Q}) \to H^*(X_z, \mathbf{Q}).$$

 $\log(r)$ is a nilpotent matrix. $\ker\log(r)$ is precisely the space of r-invariant cocycles. The main result of Deligne-Schmid-Clemens [C] is that $im(\delta)^* = \ker\log(r)$. There is a natural decomposition

(3.7)
$$H^*(X_z, \mathbf{Q}) = \ker \log(r) + im \log(r)^T.$$

 $im \log(r)^T$ is Poincaré dual to the space of vanishing cycles. Using the decomposition (3.7), we can define a map

$$(3.8) H^*(X_z, \mathbf{Q}) \to H^*(X_s, \mathbf{Q}).$$

Let $\phi: H^*(X_s, \mathbf{Q}) \to H^*(X, \mathbf{Q})$ be its composition with c^* . Our second theorem is that

Theorem 3.2 (Li-Ruan) Suppose that $T: X \to X_z$ is a small transition in complex dimension three as described previously. Then $\phi: H^*(X_z, \mathbf{R}) \to H^*(X, \mathbf{R})$ is a natural map for big quantum cohomology.

The author conjectured that

Quantum Naturality Conjecture: Suppose that $T: X \to X_z$ is a small transition in any dimension and $\phi: H^*(X_z, \mathbf{R}) \to H^*(X, \mathbf{R})$ is defined previously. Then,

- (1) $\Psi^{X_z}_{(A,k)}(\alpha,\cdots) = 0$ for $A \neq 0$ if α is Poincare dual to a vanishing cycle.i.e., $\alpha \in im \log(r)^T$.
- (2) Let $H \subset H^*(X_z, \mathbf{R})$ be a subspace such that $\phi|_H$ is injective. Then, $\phi|_H : H \to H^*(X, \mathbf{R})$ is a natural map for big quantum cohomology.

The results above are not satisfactory because they do not say anything about the the easiest transition, namely blow-down. The calculation in [RT1] (Example 8.6) shows that blow-down is not natural for quantum cohomology. But this is not the end of the story. For Calabi-Yau 3-folds, there are two other types of transitions which are not small. These are blow-down type surgeries. By the Mirror Surgery Conjecture [LR], there should be corresponding operations on the Hodge structures. The operation on the Hodge structure is obviously natural in algebraic coordinates. To compare with quantum cohomology, we should change from algebraic coordinates to the flat coordinates near the large complex structure limit (mirror transformation). This suggests

Quantum p-Naturality Conjecture: Transition induces a p-natural map on big quantum cohomology.

We hope to give shortly such an example of p-naturality in [LQR].

For the same reason, we also have

p-Quantum Minimal Model Conjecture: Any two smooth minimal models have p-isomorphic quantum cohomology

The Quantum Minimal Model Conjecture implies the p-Quantum Minimal Model Conjecture; but the converse could be false.

In Mori's program, it is essential to consider varieties with terminal singularities. It would be an important problem to work out the singular case in complex dimension three to see if the same phenomenon holds in the category of varieties with terminal singularities. Fortunately, a complete description of flops is also available in this case [K3].

4 Log Gromov-Witten invariants and degeneration formula

In this section, we describe the techniques used to prove Theorems 3.1, 3.2. There are two slightly different approaches by Li-Ruan [LR] and Ionel-Parker [IP]. Here, we present Li-Ruan's approach only. Readers should consult [IP] for their approach. Moreover, the author uses different terminology from the original paper to make it more familiar to readers who are not familiar with contact geometry. The heart of the technique is a degeneration formula for GW-invariants under semi-stable degeneration such that the central fiber has only two components. The author has already given the definition of semi-stable degeneration in the last section. In the case that the central fiber has only two components, the normal bundle of their intersection Z has opposite first Chern

classes. During the last several years, semi-stable degeneration reappeared in symplectic geometry under the names "symplectic norm sum" and "symplectic cutting" and plays an important role in some of the recent developments in symplectic geometry. The author should point out that the recent construction of symplectic geometry is independent from algebraic geometry. In fact, the symplectic construction is stronger because it shows that one can always produce a semi-stable degeneration with prescribed central fiber in the symplectic category. Of course, the analogous result is far from true in the algebraic category. There are two symplectic constructions, norm sum and cutting, which are inverse to each other. Let me present the construction of symplectic cutting due to E. Lerman [L].

Suppose that $H: M \to \mathbf{R}$ is a periodic Hamiltonian function such that the Hamiltonian vector field X_H generates a circle action. By adding a constant, we can assume that 0 is a regular value. Then $H^{-1}(0)$ is a smooth submanifold preserved by the circle action. The quotient $Z = H^{-1}(0)/S^1$ is the famous symplectic reduction. Namely, it has an induced symplectic structure.

For simplicity, we assume that M has a global Hamiltonian circle action. Once we write down the construction, we observe that a local circle Hamiltonian action is enough to define symplectic cutting.

Consider the product manifold $(M \times \mathbf{C}, \omega \oplus -idz \wedge d\bar{z})$. The moment map $H - |z|^2$ generates a Hamiltonian circle action $e^{i\theta}(x,z) = (e^{i\theta}x, e^{-i\theta}z)$. Zero is a regular value and we have symplectic reduction

(4.1)
$$\overline{M}^{+} = \{H = |z|^{2}\}/S^{1}.$$

We have the decomposition

$$\overline{M}^{+} = \{H = |z|^{2}\}/S^{1} = \{H = |z|^{2} > 0\}/S^{1} \cup H^{-1}(0)/S^{1}.$$

Furthermore,

$$\phi^+: \{H>0\} \to \{H=|z|^2>0\}/S^1$$

(4.3)
$$\phi^{+}(x) = (x, \sqrt{H(x)}).$$

is a symplectomorphism. Let

$$(4.4) M_b^+ = H^{-1}(\geq 0).$$

Then M_b^+ is a manifold with boundary and there is a map

$$(4.5) M_b^+ \to \overline{M}^+.$$

Clearly, \overline{M}^+ is obtained by collapsing the S^1 action on $H^{-1}(0)$. It is obvious that we only need a local S^1 Hamiltonian action. To obtain \overline{M}^- , we consider the circle action $e^{i\theta}(x,z)=(e^{i\theta}x,e^{i\theta}z)$ with the moment map $H+|z|^2$. $\overline{M}^+,\overline{M}^-$ are called symplectic cuttings of M. We define M_b^- similarly. By the construction, $Z=H^{-1}(0)/S^1$ with induced symplectic structure embedded symplectically into \overline{M}^\pm . Moreover, its normal bundles have opposite first Chern classes.

Symplectic norm sum is an inverse operation of symplectic cutting. Gompf developed before symplectic cutting appeared. E. Ionel observed that symplectic norm sum-symplectic cutting is the same as semi-stable degeneration such that $\overline{M}^+ \cup_Z \overline{M}^-$ is the central fiber.

The main result of this section is a degeneration formula of GW-invariants under semi-stable degeneration with the central fiber of two components or symplectic norm sum or symplectic cutting. To simplify the notation, we use the term symplectic cutting only. The first step is to introduce relative GW-invariants of a pair (M, Z) where Z is a smooth codimension two symplectic submanifold. We can always choose an almost complex structure J such that Z is an almost complex submanifold. We remark that such an almost complex structure is not generic in the usual sense. Hence, the algebraic geometer should view Z as a smooth divisor. The process of defining relative GW-invariants is similar to that of defining regular GW-invariants. First, we define relative stable maps. It is helpful to recall the definition of stable maps.

Definition 4.1 ([PW], [Ye], [KM]). A stable J-map is an equivalent class of the pair (Σ, f) . Here, Σ is a nodal marked Riemann surface with arithmetic genus g, k-marked point, and f: $\Sigma \to X$ is a continuous map whose restriction on each component of Σ (called a component of f in short) is J-holomorphic. Furthermore, it satisfies the stability condition: if $f|_{S^2}$ is constant (called ghost bubble) for some S^2 -component, the S^2 has at least three special points (marked points or nodal point).

If $h: \Sigma \to \Sigma'$ is a biholomorphic map, h acts on (Σ, f) by $h \circ (\Sigma, f) = (\Sigma', f \circ h)$. Then, $(\Sigma, f), (\Sigma', f \circ h)$ are equivalent or $(\Sigma, f) \sim (\Sigma', f \circ h)$.

Suppose that (Σ, f) is a J-stable map. A stable map can be naturally decomposed into connected components lying outside of Z (non-Z-factors) or completely inside Z (Z-factors). Both are stable maps. The division creates some marked points different from x_i . We call these marked points new marked points. In contrast, we call x_i old marked points. Obviously, different factors intersect each other at new marked points.

Definition 4.2: A label of (Σ, f) consists of (1) a division of each Z-factor into a set of stable

maps $\{f_{\mu}\}$ (call Z-subfactor) intersecting at new marked points; (2) an assignment of a nonzero integer a_p to new marked points of $\{f_{\mu}\}$, non-Z factors with the following compatibility condition.

- (1) If p is a new marked point of non-Z factor f_{η} , then $a_p > 0$ is the order of tangency of f_{η} with Z.
- (2) If p and q are new marked points where two components intersect, the $a_p = -a_q$.
- (3) If f_{μ} and f_{ν} intersect at $f_{\mu}(p_i) = f_{\nu}(q_i)$ for $1 \leq i \leq l$, then all the a_{p_i} with $1 \leq i \leq l$ have the same sign.

Let N be the projective completion of the normal bundle $E \to Z$, i.e., $N = P(E \oplus \mathbf{C})$. Then N has a zero section Z_0 and an infinity section Z_{∞} . We view Z in M as a zero section. To define relative stable map, we assign a nonnegative integer t_i to each marked point x_i such that $\sum_i t_i = [f]Z$. Denote the tuple (t_1, \dots, t_k) by T_k .

Definition 4.3: A log stable map is a triple $((\Sigma, f), T, label)$ such that each Z-subfactor f_{μ} can be lift to a stable map \tilde{f}_{μ} into N satisfying (1) \tilde{f}_{μ} intersects Z_0, Z_{∞} at marked points (old or new) only; (2) If p_i is a new marked point on \tilde{f}_{μ} , then \tilde{f}_{μ} intersects Z_0 (resp. Z_{∞}) at p_i with order $|a_{p_i}|$ if $a_{p_i} > 0$ (resp. $a_{p_i} < 0$). (3) Let x_i be an old marked point. If x_i is on a non-Z factor f_{η} , then f_{η} intersects Z at x_i with order t_i . If x_i is on a Z-subfactor f_{μ} , then \tilde{f}_{μ} intersects Z_0 at x_i with order t_i .

It is easy to show that if the lifting \tilde{f}_{μ} exists, it is unique up to the complex multiplication on the fiber of N. Let $\overline{\mathcal{M}}_{A}^{M,Z}(g,T_{k},J)$ be the moduli space of relative stable maps with fixed T. Clearly, there is a map

(4.6)
$$\pi: \overline{\mathcal{M}}_{A}^{M,Z}(g, T_{k}, J) \to \overline{\mathcal{M}}_{A}^{X}(g, k, J).$$

Now let's explain the motivation of above definition. Consider the convergence of a sequence of J-map (Σ_n, f_n) . Of course, f_n will converge to a stable map (Σ, f) . In general, (Σ, f) may have some Z-factors. Recall that the puncture disc $D - \{0\}$ is biholomorphic to half cylinder $S^1 \times [0, \infty)$. Now, we do this fiberwisely over Z. We can view M - Z as an almost complex manifold with an infinite long cylinder end. We call it cylindric model. Now, we reconsider the convergence of (Σ_n, f_n) in the cylindric model. The creation of a Z-subfactor f_μ corresponds to disappearance of part of $im(f_n)$ into the infinity. We can use the translation to rescale back missing part of $im(f_n)$.

In the limit, we obtain a stable map \tilde{f}_{μ} into $\tilde{Z} \times \mathbf{R}$, where \tilde{Z} is the circle bundle consisting the united vectors of E. N is just the closure of $\tilde{Z} \times \mathbf{R}$. One can further show that \tilde{f}_{μ} indeed is a stable map into N. Therefore, we obtain a lifting of f_{μ} . The label is used to specify the lifting.

Suppose that $t_i = 0$ for $i \leq l$ and $t_i > 0$ for i > l. We have evaluation maps

for $i \leq l$, and

$$\Xi_i^M : \overline{\mathcal{B}}_A^{X,Z}(g, T_k, J) \to Z$$

for any j > l. Let $\alpha_i \in H^*(M, \mathbf{R}), \beta_j \in H^*(Z, \mathbf{R})$.

Roughly, the log GW-invariants are defined as

$$(4.10) \qquad \Psi_{(A,g,T_k)}^{(M,Z)}(K;\alpha_1,\cdots,\alpha_l;\beta_{l+1},\cdots,\beta_k) = \int_{\overline{\mathcal{M}}_A^{M,Z}(g,T_k,J)} \chi_{g,k}^* K \wedge \Xi_{g,k}^* \prod_i \alpha_i \wedge P^* \prod_i \beta_j.$$

To be precise, the virtual techniques developed by Fukaya-Ono [FO], Li-Tian[LT3], Ruan[R5], Siebert [S] apply to this case. For example, we can construct a virtual neighborhood (7.1 [LR]) of $\overline{\mathcal{M}}_A^{M,Z}(g,T_k,J)$. Then we integrate the integrand (1.17) over the the normalization of virtual neighborhood.

Clearly, there is a map

$$(4.11) R: \overline{\mathcal{M}}_A^{M,Z}(g,T_k,J) \to \overline{\mathcal{M}}_A^M(g,k,J).$$

However, it may not be surjective even if $T_k = (1, \dots, 1)$. Because of this, we remark that when $T_k = \{1, \dots, 1\}$, $\Psi^{(M,Z)}$ is usually different from the ordinary or absolute GW-invariant. So $\Psi^{(M,Z)}$ is not a "generalized" GW-invariant.

Theorem 4.3 (Li-Ruan)(Theorem 7.6 [LR]):

- (i). $\Psi^{(M,Z)}_{(A,g,T_k)}(K;\alpha_1,...,\alpha_l;\beta_{l+1},...,\beta_k)$ is well-defined, multilinear and skew-symmetric.
- (ii). $\Psi^{(M,Z)}_{(A,g,T_k)}(K;\alpha_1,...,\alpha_l;\beta_{l+1},...,\beta_k)$ is independent of the choice of forms K,α_i,β_j representing the cohomology classes $K,[\beta_j],[\alpha_i]$, and independent of the choice of virtual neighborhoods.
- (iii). $\Psi^{(M,Z)}_{(A,g,T_k)}(K;\alpha_1,\ldots,\alpha_l;\beta_{l+1},\ldots,\beta_k)$ is independent of the choice of almost complex structures on M where Z is an almost complex submanifold, and hence is an invariant of (M,Z).

If K = 1, we will drop K from the formula. This invariant is called a *primitive log GW-invariant*. In this article, we will give a degeneration formula for primitive invariants and comment on how to modify it for non-primitive invariants.

Now we explain the degeneration formula of GW-invariants under symplectic cutting. First of all, symplectic cutting defines a map

$$(4.12) \pi: M \to \overline{M}^+ \cup_Z \overline{M}^-,$$

where the right hand side is the central fiber of the degeneration. By construction, $\omega^+|_Z = \omega^-|_Z$. Hence, the pair (ω^+, ω^-) defines a cohomology class of $\overline{M}^+ \cup_Z \overline{M}^-$, denoted by $\omega^+ \cup_Z \omega^-$. It is easy to observe that

$$(4.13) \pi^*(\omega^+ \cup_Z \omega^-) = \omega.$$

 π induces a map

(4.14)
$$\pi_*: H_2(M, \mathbf{Z}) \to H_2(\overline{M}^+ \cup_Z \overline{M}^-, \mathbf{Z}).$$

Let $B \in \ker(\pi_*)$. By (4.13), $\omega(B) = 0$. Define $[A] = A + \ker(\pi_*)$ and

(4.15)
$$\Psi^{M}_{([A],g,k)} = \sum_{B \in [A]} \Psi_{(B,g,k)}.$$

For any $B, B' \in [A]$, $\omega(B) = \omega(B')$. By the Gromov compactness theorem, there are only finitely many such B represented by J-holomorphic stable maps. Therefore, the summation in (4.15) is finite. Moreover, the cohomology class α_i is not arbitrary. Namely, let $\alpha_i^{\pm} \in H^*(\overline{M}^{\pm}, \mathbf{R})$ such that $\alpha_i^+|_Z = \alpha_i^-|_Z$. It defines a class $\alpha_i^+ \cup_Z \alpha_i^- \in H^*(\overline{M}^+ \cup_Z \overline{M}^-, \mathbf{R})$. We choose $\alpha_i = \pi^*(\alpha^+ \cup_Z \alpha^-)$. The degeneration formula is a big summation

(4.16)
$$\Psi^{M}_{([A],g,k)}(\alpha_1,\cdots,\alpha_k) = \sum_{indices} terms.$$

Next, we give a procedure to write down the terms on the right hand side.

Step (1). We first write down a graph representing the topological type of a degenerate Riemann surface in $\overline{M}^+ \cup_Z \overline{M}^-$ with the following properties: (i) Each component is completely inside either \overline{M}^+ or \overline{M}^- ; (ii) the components only intersect each other along Z. (iii) No two components where both in \overline{M}^+ or in \overline{M}^- intersect each other; (iv) The arithmetic genus is g and the number of marked points is k; (v) The total homology class is $\pi_*([A])$. (vi). Each intersection point carries a positive integer representing the order of tangency. We use C to denote such a graph. C will be used as the index of summation and (4.16) can be written as

(4.17)
$$\Psi^{M}_{([A],g,k)}(\alpha_1,\cdots,\alpha_k) = \sum_{C} \Psi_{C}.$$

Step (2). Suppose that C has components $C_1, \dots C_s$ and let (A_{C_i}, g_{C_i}) be the homology class and genus of the C_i -component. Then,

(4.18)
$$\Psi_C = r_C \prod_i \Psi_{(A_{C_i}, g_{C_i}, T_i)}^{(\overline{M}^?, Z)}(variables),$$

where $\overline{M}^{?}$ is the one of \overline{M}^{\pm} in which C_i lies and r_C is the product of certain numbers to be determined. T_i is given by the order of tangency. Here, the original marked points have zero order of tangency.

Step(3). The following two steps determine the variables of each term in (4.18) and r_C . If a marked point x_i appears in some component C_t , then α_i^{\pm} should be in the variable of $\Psi_{(A_{C_t}, g_{C_t}, T_t)}^{(\overline{M}^?, Z)}(variables)$. \pm depends on whether C_t lies in \overline{M}^+ or \overline{M}^- .

Step(4). $\Psi_{(A_{C_i},g_{C_i})}^{(\overline{M}^i,Z)}(variables)$ has other variables as well associated to intersection points. Suppose that y is an intersection point of C_i and C_j components with order of tangency k_y . Let β_a be a basis of $H^*(Z,\mathbf{R})$ and $\eta^{ab} = \int_Z \beta_a \beta_b$. Let (η_{ab}) be its inverse, which can be thought of as the intersection matrix of the Poincaré dual of β_a, β_b . Then y contributes a term β_a in $\Psi_{(A_{C_i},g_{C_i})}$, β_b in $\Psi_{(A_{C_i},g_{C_i})}$ and $k_y\eta_{ab}$ in r_C .

The four steps above will completely determine the formula (4.16).

Theorem 4.1 (Li-Ruan([LR], Theorem 7.9,7.10)) There is a degeneration formula of GW-invariants under symplectic cutting described by Step (1)-Step (4).

An important comment is that different α_i^{\pm} may define the same α_i . Then we have different ways to express $\Psi^M(\dots,\alpha_i,\dots)$. This is very important in the application of the degeneration formula (4.16). For example, The Pioncaré dual of a point can be chosen to have its support completely inside \overline{M}^+ or \overline{M}^- .

To derive a degeneration formula with class K, let us consider its Poincaré dual K^* . Then we have to specify a degeneration K_{∞}^* of K^* and look at what kind of graph C could appear in the degeneration of K^* . Namely, we look for those C for which we obtain an element of K_{∞}^* after we contract all the unstable components of C. For example, suppose that $K^* = \{\Sigma\}$. We can require that Σ stay in the interior of $\overline{M}_{g,k}$. This requirement will force C to have one component the same as Σ in one of \overline{M}^{\pm} and other components being unstable rational components.

Using Theorem 4.1, we can prove Theorem 3.1,3.2. Consider a simple flop $X \rightsquigarrow \tilde{X}$. They have the same blow-up X_b . Take a trivial family of X and blow up the central fiber along the the rational curve L. This is a semi-stable degeneration whose central fiber is a union of X_b and Y_L , where Y_L is the projective completion of normal bundle O(-1) + O(-1) of L intersecting with X_b along the projectivization of the normal bundle $P(O(-1) + O(-1)) \cong \mathbf{P}^1 \times \mathbf{P}^1$. The crucial information for the proof of Theorem 3.1 is that $C_1(Y_L) = 3Z_{\infty}$. An index calculation shows that the only nonzero term in (4.16) is that the C lies completely in X_b or Y_L . We repeat the same argument for \tilde{X} to express the invariant of \tilde{X} in terms of that of X_b, Y_L . We then obtain a formula for the change of GW-invariants under a simple flop. To conclude the isomorphism of quantum cohomology, we need the amazing fact that the contribution of multiple cover maps to L cancels the error from the change of the classical triple product under a simple flop.

To prove Theorem 3.2, we consider the case when X_s has only a ordinary double point. We then do the semi-stable reduction to obtain a semi-stable degeneration. In this case, the central fiber has again only two components X_b, Y_s (a fact I learnt from H. Clemens), where X_b is the blow-up along L and Y_s is a quadratic 3-fold intersecting with X_b along a hyperplane H. This is completely consistent with the symplectic cutting description in section 2 of [LR]! Note that the first Chern class of a quadric 3-fold is 3H! This is precisely what we need in the case of flop. Using our degeneration formula, we conclude that the only nonzero term in (4.16) is the case that C stays in either X_b, Y_s . But there is no holomorphic curve in Y_s not intersecting H. Hence, it must be in X_b . Therefore, we get an expression (Theorem 8.1,[LR]) of the invariant of X_z in terms of the relative invariant of X_b and hence X by the previous argument. The rest of the proof is just sorting out the formula for quantum 3-fold points.

For more general flops or small transitions, an argument of P. Wilson shows that one can reduce it to the above case by an almost complex deformation. Hence, our previous argument works in all cases. If the reader is looking for an algebraic proof of our theorem, it seems that one can generalize our degenerations formula to semi-stable degeneration with an arbitrary number of components. Then one can use this more general degeneration formula to bypass the almost complex deformation. It is clear that such a general degeneration formula should also contain relative GW-invariants, but the combinatorics will be much more complicated.

5 Symplectic minimal model program

After going over technical mathematics in the last section, it is time to have some fun and make some wild speculations. Mori's minimal model program can be viewed as a surgery theory of contractions and flip-flops. As we mentioned in section three, transitions play a crucial role in the classification

of Calabi-Yau 3-folds. The author hopes that he has convinced the reader that they also play a crucial role in quantum cohomology. One essence of symplectic geometry is the flexibility to deform complex structure. Therefore, transition is also a natural operation in symplectic geometry. It is natural that we speculate that there should be a minimal model theory with transitions as fundamental surgeries. We call this proposed theory the "symplectic minimal model program". The holomorphic version of symplectic minimal model program is no longer in the category of birational geometry. Instead, it addresses the important problem of connecting different moduli spaces of complex manifolds. An interesting question is: what are the minimal models in the holomorphic symplectic minimal model program? Recall that in Mori theory a minimal model has nef K_X . We speculate that a minimal model in the moduli minimal model program should be either Pic(X) = 1 or K_X is ample. Like the case of minimal models program, there are exceptional cases. It is of great interest to study these exceptional cases as well. Clearly, transition improves singularities. The author does not have any feeling for what kind of singularities should be allowed in the holomorphic minimal model program.

In my view, the guiding problem of symplectic geometry should be the classification of symplectic manifolds. Symplectic minimal model program should also be viewed as a classification scheme of symplectic manifolds. In topology, it is rare that we can label manifolds. An essential step is to establish some fundamental surgeries and classify a class of manifolds under such surgeries. These fundamental surgeries should simplify the manifolds. In dimension two, the fundamental surgery is connected sum. In dimension three, the fundamental surgeries are connected sums over spheres and tori. In dimension 4, people are still struggling to understand the fundamental surgeries. A natural question is "what are fundamental surgeries of symplectic manifolds?" I believe that transition is one of the fundamental surgeries of symplectic manifolds. In fact, any surgeries which are natural with respect to quantum cohomology deserve our attention, if we believe that quantum cohomology is a fundamental invariant of symplectic manifolds.

Symplectic minimal model program also provide a scheme to attack Arnold conjectures. Recall that the Arnold conjecture for degenerate hamiltonian is still an open problem. The obstruction is precisely the existence of holomorphic curves. The symplectic minimal model program is designed to kill all the holomorphic curves. Therefore, Arnold conjectures should hold for minimal model. The rest is to do transition in Hamiltonian invarant fashion.

This line of thought is very appealing because transition has a beautiful interpretation in term of classical symplectic geometry.

In a neighborhood of a singularity, the boundary has a natural contact structure. It is known classically that there are two ways to "fill" a contact manifold. A contact manifold could bound a resolution of a singularity or a neighborhood of the zero section of a cotangent bundle. In the case of a singular point, transition is a local duality which interchanges these two fillings. In the general case, we need to consider a fiber-wise version of the above construction.

Therefore, it is natural to consider a "symplectic minimal model program" using transition as the fundamental surgery. As the author showed in [R3], one can generalize the Mori cone NE(X) to symplectic manifolds. Transition simplifies a symplectic manifold X in the sense that it simplifies the Mori cone NE(X). One important ingredient in Mori's program is the interpolation between the Mori cone and the ample cone, which are related by $NE(X) = \overline{K(X)^*}$. In symplectic geometry, we no longer have such a relation. So we encounter severe difficulties at the first step of our symplectic minimal model program. It is probably a long shot to establish such a program. But I have no doubt that much interesting mathematics will come out of our investigation. We end our discussion with following question: What is the minimal model in the symplectic minimal model program?

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